ISRAEL JOURNAL OF MATHEMATICS 78 (1992), 131–142

SIMPLE SYSTEMS AND THEIR HIGHER ORDER SELF-JOININGS

BY

E. GLASNER

School of Mathematics Tel Aviv University
Tel Aviv, 69978, Israel

AND

B. HOST

Dept. Math. Info. Universite d'Aix-Marseille II
Case 901, 163, avenue de Luminy-13288, Marseille Cedex 9, France

AND

D. RUDOLPH

Department of Mathematics, University of Maryland
College Park, MD 20742, U.S.A.

ABSTRACT

The purpose of this work is to study the joinings of simple systems. First the joinings of a simple system with another ergodic system are treated; then the pairwise independent joinings of three systems one of which is simple. The main results obtained are: (1) A weakly mixing simple system with no non-trivial factors with absolutely continuous spectral type is simple of all orders. (2) A weakly mixing system simple of order 3 is simple of all orders.

1. Introduction

Given three ergodic systems (measure preserving transformations) and an ergodic pairwise independent joining σ of the three, it is a basic problem in ergodic theory to find conditions under which σ is independent (see e.g. [H]). We treat here a special case of this general problem.

Received April 14, 1991 and in revised form December 3, 1991

Let $k \geq 1$ be an integer, a k-fold self-joining σ of an ergodic system (X, μ, T) (i.e. σ is a measure on a product of k copies of X denoted $X_1, X_2, \ldots, X_k$ invariant under the product transformation and projecting onto μ in each coordinate) is called an **off-diagonal** if σ is the image of μ under the map $x \mapsto (\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x))$ of X into $\Pi_{i=1}^k X_i$, where each φ_i is an element of the group $C(X)$ of automorphisms of (X, μ, T). σ is a **product of off-diagonals** if there exists a partition $(J_1, \ldots, J_m)$ of $\{1, \ldots, k\}$ such that

(i) For each ℓ, the projection of σ on $\Pi_{i \in J_\ell} X_i$ is an off-diagonal

(ii) The systems $(\Pi_{i \in J_\ell} X_i,\ 1 \leq \ell \leq m)$ are independent.

(X, μ, T) is **simple of order** k (or k-**simple**) if every k-fold ergodic self-joining of X is a product of off-diagonals. We say that it is **simple** when it is simple of order two. (See [R], [V] and [J–R,1]).

Recently, J. King [K] has shown that when (X, μ, T) is weakly mixing and simple of order 4 then it is simple of all orders. In this work we show, using different methods, that in fact weak mixing (w.m.) and simple of order 3 imply simple of all orders (Theorem 5). If in addition to weak mixing we assume that (X, μ, T) admits no non-trivial factors with absolutely continuous spectral measure then simple implies k-simple for all $k \geq 2$ (Theorem 4). In particular, this is the case when (X, μ, T) is rigid (in the sense of [F–W]).

The main lemma (Lemma 2) used in the proofs of Theorems 4 and 5, deals with the following situation. We are given a 3-fold ergodic, pairwise independent joining σ of a simple system (X, μ, T), an ergodic system (Y, ν, T) and a weakly mixing system (Z, λ, T). The lemma shows that if for some $n \neq 0$ the joinings σ and $\sigma_n = (id \times id \times T^n)\sigma$ are not orthogonal over $Y \times Z$ (see definition below), then σ is the independent joining i.e. the product measure $\mu \times \nu \times \lambda$. (Actually the statement is slightly stronger; also see [M] for a topological version of this lemma). It is then shown (Theorem 3) that under these circumstances when σ is not independent, (X, μ, T) admits a non-trivial factor with absolutely continuous (with respect to Lebesgue measure on $\mathbf{T}$) spectral type.

Here are some conventions we use. When denoting a measure preserving system the corresponding σ-algebra is omitted. All sets and functions that appear are assumed measurable. We denote by T the acting transformation in every system considered, with the exception of product systems where $T \times T$ or $T \times T \times T$ etc. are used. $C(X)$ will denote the group of measurable automorphisms of the system (X, μ, T); Δ_μ will denote the image of μ under the map $x \mapsto (x, x)$ of X, onto the diagonal of $X \times X$ (and similarly for higher orders). For $x \in X$, δ_x is the point mass at x. The map $\mathrm{proj}_X\, X \times Y \to X$ is the projection on the X coordinate and similarly for Y.

In section 2 some basic notations and results on 2-fold joinings are given. With the exception of Lemma 1 these results can be considered well known. In section 3 the joinings of a simple system with an ergodic system are studied (Theorems 1 and 2). These results are closely related to results in [J–R,1] especially Theorem 4.1 of this paper. The basic Lemma 2 and Theorems 3 and 4 are proved in section 4, while Theorem 5 is proved in the last section.

The history of this paper is somewhat complicated. Two of the authors (E.G. and D.R.) first wrote a paper containing some of the results of the present one (mainly Lemma 2, the simplicity of all orders for rigid simple systems, and an observation on the existence of infinitely many pairwise independent factors related to the simple system in case it is not 3-simple).

The third author (B.H.), independently, obtained all the results of the present work. When we learned about the existence of each others works, the original paper was withdrawn and the present one, incorporating the two works, was written.

2. Joinings of Two Systems

Given a joining σ of the systems (X,μ,T) and (Y,ν,T) we denote by σ^* the corresponding joining of (Y,ν,T) and (X,μ,T); thus σ is a measure on $X \times Y$ whereas σ^* is defined on $Y \times X$. The **disintegration** of σ over (Y,ν) is the representation $\sigma = \int \sigma_y \times \delta_y \, d\nu(y)$ where $y \mapsto \sigma_y$ is a measurable T-equivariant map of Y into the space of probability measures on X, such that $\mu = \int \sigma_y d\nu(y)$. We denote by $E_\sigma : L^2(X,\mu) \to L^2(Y,\nu)$ the conditional expectation operator given by

$$E_\sigma f(y) = \int f(x) d\sigma_y(x) \qquad \text{for } \nu\text{-a.e. } y.$$

Equivalently E_σ is defined by

$$\int E_\sigma f(y) g(y) d\nu(y) = \int f(x) g(y) d\sigma(x,y)$$

$(f \in L^2(X,\mu), g \in L^2(Y,\nu))$. It is easy to check that $E_{\sigma^*} : L^2(Y,\nu) \to L^2(X,\mu)$ is the adjoint of E_σ. Let $L_0^2(X,\mu)$ be the subspace of $L^2(X,\mu)$ consisting of functions of zero integral. Then σ is independent (i.e. $\sigma = \mu \times \nu$) iff the restriction of E_σ to $L^2_0(X,\mu)$ is the zero operator.

Now let τ be a joining of (Y,ν,T) and (Z,λ,T). Define the (conditionally independent) **product of σ and τ over** (Y,ν,T), denoted $\sigma \underset{Y}{\times} \tau$, to be the measure on $X \times Y \times Z$ given by

$$\sigma \underset{Y}{\times} \tau = \int \sigma_y \times \delta_y \times \tau^*_y d\nu(y)$$

or equivalently by the equation

$$\int f(x)g(y)h(z)d\sigma \underset{Y}{\times} \tau = \int E_\sigma f(y) \cdot g(y) \cdot E^*_\tau h(y) d\nu(y)$$

(f, g, h bounded on X, Y and Z respectively).

The measure on $X \times Z$ given by

$$\tau \circ \sigma = \int \sigma_y \times \tau^*_y d\nu(y) = \mathrm{proj}_{X \times Z}(\sigma \underset{Y}{\times} \tau)$$

is called the **composition** of σ and τ. It is easy to check that $E_{\tau\circ\sigma} = E_\tau E_\sigma$. The joinings σ of (X,μ,T) and (Y,ν,T), and τ^* of (Z,λ,T) and (Y,ν,T) are said to be **orthogonal** or **independent relative to** Y if $\tau\circ\sigma$ is the product measure $\mu \times \lambda$. The joining σ is orthogonal to itself relative to Y iff it is independent.

Suppose now σ is an ergodic joining (i.e. the system $(X \times Y, \sigma, T \times T)$ is ergodic), then either for ν-a.e. y, σ_y is a continuous measure or there exists a positive integer r such that for ν-a.e. y, σ_y is an atomic measure equidistributed on a finite subset of X of cardinality r. In the latter case $(X \times Y, \sigma, T \times T)$ is an r to one extension of (Y,ν,T) and we say that σ is of **finite type** (or more precisely of **type** r).

LEMMA 1: *Let σ and τ be two ergodic joinings of the ergodic systems (X,μ,T) and (Y,ν,T) and let $\Delta \subset X \times X$ be the diagonal. Then $\tau^* \circ \sigma(\Delta) > 0$ iff $\tau = \sigma$ and σ is of finite type. In this case $\tau^* \circ \sigma(\Delta) = \frac{1}{r}$ where r is the type of σ.*

Proof: Since

$$\tau^* \circ \sigma(\Delta) = \int \sigma_y \times \tau_y(\Delta) d\nu(y) = \int \sum \sigma_y\{x\}\tau_y\{x\} d\nu(y)$$

it follows that when $\tau = \sigma$ and this joining is of finite type r, then $\tau^* \circ \sigma(\Delta) = r\frac{1}{r^2} = r$. Conversely suppose $\tau^* \circ \sigma(\Delta) > 0$, then the above formula shows that both σ and τ are of finite type, say s and t respectively, and that

$$\tau^* \circ \sigma(\Delta) = \frac{1}{t \cdot s} \int card(A_y \cap B_y) d\nu(y) \ ,$$

where A_y and B_y are the finite supports of σ_y and τ_y respectively. Since $TA_y = A_{T_y}$ and $TB_y = B_{T_y}$ ν-a.e. it follows that the subset $E = \{(x,y) : x \in A_y \cap B_y\}$ of $X \times Y$ is $T \times T$ invariant. Since moreover

$$\sigma(E) = \int \sigma_y \times \delta_y(E) d\nu(y) = \frac{1}{s} \int card(A_y \cap B_y) d\nu(y)$$
$$= t(\tau^* \circ \sigma)(\Delta) > 0 ,$$

the ergodicity of σ implies $\sigma(E) = 1$. Thus card $(A_y \cap B_y) = s$, whence $A_y \subset B_y$ for ν-a.e. y. By symmetry also $B_y \subset A_y$ and we conclude that $\sigma = \tau$ and that σ is of type $r = s = t$. ∎

3. Joinings of a Simple System and an Ergodic One

For the rest of this paper we shall assume that (X, μ, T) is a simple system.

THEOREM 1: *Let σ and σ' be two ergodic joinings of (X, μ, T) and (Y, ν, T), where (X, μ, T) is simple and (Y, ν, T) ergodic. Then either σ and σ' are orthogonal over Y or there exists $\varphi \in C(X)$ such that $\sigma' = (\varphi \times id)\sigma$.*

Proof: Let $\gamma = \sigma \underset{Y}{\times} \sigma'$ be considered as a measure on $X \times Y \times X'$ where X' is a copy of X. Let $\gamma = \int_\Omega \omega dP(\omega)$ be the ergodic decomposition of γ. The elements of Ω are ergodic joinings of X, Y and X'. Let $\Omega_0 \subset \Omega$ be the subset of those $w \in \Omega$ for which the projection of ω on $X \times X'$ is not the product measure $\mu \times \mu'$. Clearly $P(\Omega_0) = 0$ implies $\sigma'^* \circ \sigma = \mu \times \mu$; i.e. σ and σ' are orthogonal over Y.

Assume now $P(\Omega_0) > 0$; then for P-a.e. $\omega \in \Omega_0$ we have

(i) the projection of ω on $X \times X'$ is an ergodic joining $\neq \mu \times \mu'$, hence of the form $\Delta_\mu^\varphi = id \times \varphi(\Delta_\mu)$, where $\varphi = \varphi_\omega \in C(X)$ is considered as an isomorphism of X onto X'.

(ii) the projections of ω on $X \times Y$ and $Y \times X'$ are σ and σ'^* respectively.

Write $\omega = \int \delta_y \times \omega_y d\nu(y)$ where ω_y is a measure on $X \times X'$, then it follows that $\int \omega_y d\nu(y) = \Delta_\mu^\varphi$ and we conclude that for ν-a.e. y, ω_y is supported on $\Delta^\varphi = (id \times \varphi)(\Delta)$. Now

$$\sigma = \int (\text{proj}_X \, \omega_y) \times \delta_y d\nu(y) \quad \text{and}$$
$$\sigma'^* = \int \delta_y \times (\text{proj}_{X'} \, \omega_y) d\nu(y)$$

imply that $\sigma' = (\varphi \times id)\sigma$. ∎

THEOREM 2: *Let σ be a non independent ergodic joining of the simple system (X,μ,T) and the ergodic (Y,ν,T). Then there exists a positive integer r and a compact subgroup K of $C(X)$ such that for the corresponding group factor $(X,\mu,T)\xrightarrow{\pi}(U,\rho,T)\cong(X/K,\mu,T)$ we have*

(i) *σ is the relatively independent product of (X,μ,T) and $(U\times Y,\tau,T\times T)$ over (U,ρ,T) where τ is the image of σ under $\pi\times id$.*

(ii) *τ is a joining of finite type r and for every $f,g\in L_0^2(U,\rho)$*

$$\int E_\tau f(y)E_\tau g(y)d\nu(y)=\frac{1}{r}\int f(u)g(u)d\rho(u)\ .$$

Proof: We use the notations introduced in the proof of theorem 1, and let $\sigma=\sigma'$. Since σ is not orthogonal to itself we conclude that $P(\Omega_0)>0$. Let K be the set of those $\varphi\in C(X))$ for which $(\varphi\times id)\sigma=\sigma$. Clearly K is a closed subgroup of $C(X)$ and it is not hard to see that $P\mid_{\Omega_0}$ induces a finite K-invariant measure on K. A theorem of A. Weil now implies that K is compact (the topology is that of convergence in measure, see [V]). Denote by $(X,\mu,T)\xrightarrow{\pi}(U,\rho,T)$ the quotient map and quotient system obtained modulo K, and let τ be the image of σ under $\pi\times id$. If $\tau=\int\delta_u\times\tau_u d\rho(u)$ is the disintegration of τ over (U,ρ) then for $f\in L^2(X,\mu),\ g\in L^2(Y,\nu)$ and $\varphi\in K$

$$\int f\otimes g d\sigma=\int f(x)g(y)d\sigma(x,y)=\int(f\circ\varphi)(x)g(y)d\sigma(x,y)\ ;$$

hence denoting by m the Haar measure on K

$$\begin{aligned}\int f\otimes g d\sigma&=\iint(f\circ\varphi)(x)g(y)d\sigma(x,y)dm(\varphi)\\&=\int Ef(u)g(y)d\tau(u,y)\\&=\int(\int f(\varphi x)dm(\varphi))(\int g(y)d\tau_u(y))d\rho(u)\\&=\int f\otimes g\ d\mu\underset{U}{\times}\tau\ ,\end{aligned}$$

where $E:L^2(X,\mu)\to L^2(U,\rho)$ is the conditional expectation operator. Thus $\sigma=\mu\underset{U}{\times}\tau$ and (i) is proved.

Since, with obvious notations, for P-a.e. $\omega\in\Omega\backslash\Omega_0$, $\mathrm{proj}_{U\times U'}\omega=\rho\times\rho'$ and $\mathrm{proj}_{U\times U'}\omega=\Delta_\rho$ for $\omega\in\Omega_0$ ($\sigma=\sigma'$ implies that φ_ω , in (i) in the proof of theorem 1, must be the identity.) we get:

$$\tau'^* \circ \tau = \mathrm{proj}_{U\times U'}\gamma = \int \mathrm{proj}_{U\times U'}\,\omega dP(\omega)$$
$$= P(\Omega_0)\Delta_\rho + (1-P(\Omega_0))\rho\times\rho' \ .$$

Since $P(\Omega_0) > 0$, lemma 1 yields the fact that τ is of finite type r where $P(\Omega_0) = \frac{1}{r}$. Moreover for $f,g \in L_0^2(U,\rho)$

$$\int E_\tau f(y)\cdot E_\tau g(y) = \int f(u)g(u')d\tau'^* \circ \tau(u,u')$$
$$= \frac{1}{r}\int f(u)g(u)d\rho(u) \ . \qquad \blacksquare$$

4. Pairwise Independent Joinings

The situation considered in this section is that of an ergodic joining σ of three systems (X,μ,T), (Y,ν,T) and (Z,λ,T) of which the first is simple, the second ergodic and the third weakly mixing. We first have the following corollary of theorem 2.

COROLLARY: *If Y and Z are independent but X and Y are not, then $X\times Y$ and Z are independent.*

Proof: By Theorem 2, $(X\times Y,\theta,T\times T)$ is a distal extension of (Y,ν,T). On the other hand since (Z,λ,T) is w.m. the extension $(Y\times Z,\nu\times\lambda,T\times T)\to(Y,\nu,T)$ is relatively weakly mixing. By [Fu] these two extensions are relatively disjoint over Y; i.e. $\sigma = \theta \underset{Y}{\times} (\nu\times\lambda) = \theta\times\lambda$. $\blacksquare$

LEMMA 2: *Let σ be an ergodic joining of X,Y and Z for which X and Z as well as Y and Z are independent. If for some $n\neq 0$, σ and $\sigma_n = (id\times id\times T^n)\sigma$ are not orthogonal relative to $Y\times Z$ (i.e. $\sigma_n^*\circ\sigma \neq \mu\times\mu$), then $X\times Y$ and Z are also independent.*

Proof: By the corollary we may assume that also X and Y are independent. Suppose that for $n\neq 0$, σ and σ_n are not orthogonal over $(Y\times Z,\nu\times\lambda,T\times T)$, then according to theorem 1 there exists $\varphi\in C(X)$ such that $\sigma_n = (\varphi\times id\times id)\sigma$ and it follows that $F\sigma = \sigma$ for $F = \varphi\times id\times T^{-n}$.

Let f be a function on Y; for every function h on $X\times Z$:

$$\int E_\sigma f(x,z)h(x,z)d\mu(x)d\lambda(z) = \int f(y)h(x,z)d\sigma(x,y,z)$$

$$= \int f(y)h(x,z)dF\sigma(x,y,z) = \int f(y)h(\varphi x, T^{-n}z)d\sigma(x,y,z)$$

$$= \int E_\sigma f(x,z)h(\varphi x, T^{-n}z)d\mu(x)d\lambda(z) = \int E_\sigma f(\varphi^{-1}x, T^n z)h(x,z)d\mu(x)d\lambda(z)$$

and $E_\sigma f$ is a $\varphi^{-1} \times T^n$ invariant function.

Since (Z,λ,T) is weakly mixing $E_\sigma f$ depends only on x. This means that $E_\sigma f$ is the conditional expectation of f on X, which by independence of X and Y is a constant. We thus have shown that $X \times Z$ and Y are independent and the proof is complete. ∎

Notice that in this lemma the assumption that Y and Z are σ-independent was used tacitly when it was assumed that σ_n is a joining of X and $Y \times Z$. For this assumption means that the $Y \times Z$ projection of σ is $id \times T^n$ invariant. This is the case iff Y and Z are independent.

THEOREM 3: *Let (X,μ,T) be simple, (Y,T,ν) ergodic and (Z,λ,T) weakly mixing. If there exists an ergodic pairwise independent joining σ of the three systems which is not independent then (X,μ,T) admits a non-trivial factor whose maximal spectral type is absolutely continuous with respect to Lebesgue measure m on $\mathbf{T}$.*

Proof: We consider σ as a joining of X and $Y \times Z$. Let (U,ρ,T) be the factor of (X,μ,T) whose existence is proved in theorem 2 and let τ be the projection of σ on $U \times Y \times Z$. Since U is non-trivial and since X and $Y \times Z$ are relatively independent over U, it follows that τ is not independent. On the other hand τ is clearly pairwise independent. We are going to show that the maximal spectral type of (U,ρ,T) is absolutely continuous. For $n \in \mathbf{Z}$ let $\tau_n = (id \times id \times T^n)\tau$; then lemma 2 implies that for $n \neq 0$, $\tau_n^* \circ \tau = \rho \times \rho$. Let $f \in L_0^2(U,\rho)$, we need to show that the correlation measure α_f corresponding to f (i.e. the measure on $\mathbf{T}$ whose Fourier coefficients are given by $\widehat{\alpha}_f(k) = \int f(T^k u)\overline{f}(u)d\rho(u) \quad (k \in \mathbf{Z})$), is absolutely continuous with respect to m.

Let $F = E_\tau f \in L^2(Y \times Z, \nu \times \lambda)$ and let ω be its correlation measure for the $\mathbf{Z}^2$ action on $Y \times Z$; i.e. ω is the positive measure on $\mathbf{T}^2$ whose Fourier coefficients are given by

$$\widehat{\omega}(p,q) = \int F(T^p y, T^q z)\overline{F}(y,z)d\nu(y)d\lambda(z) \quad (p,q \in \mathbf{Z})$$

Let $\tau = \int \tau_{(y,z)} d\nu(y) d\lambda(z)$ be the disintegration of τ over $Y \times Z$, then for $(p,q) \in \mathbf{Z}^2$ we have

$$\begin{aligned} F(T^p y, T^q z) &= E_\tau f(T^p y, T^q z) \\ &= \int f(x) d(\tau_{(T^p y, T^q z)})(x) \\ &= \int f(x) d\tau_{(T\times T)^p (y, T^{q-p} z)}(x) \\ &= \int f(x) d(T \times T)^p \tau_{(y, T^{q-p} z)}(x) \\ &= \int f(T^p x) d\tau_{(y, T^{q-p} z)}(x) = E_{\tau_{q-p}} (T^p f)(y,z). \end{aligned}$$

Let θ be the image of ω under the map $(s,t) \mapsto s+t$ of $\mathbf{T}^2$ onto $\mathbf{T}$. Then for $p \in \mathbf{Z}$ we have $\widehat{\theta}(p) = \widehat{\omega}(p,p)$ and by theorem 2

$$\begin{aligned} \widehat{\theta}(p) = \widehat{\omega}(p,p) &= \int E_\tau (T^p f)(y,z) E_\tau \overline{f}(y,z) d\nu(y) d\lambda(z) \\ &= \frac{1}{r} \int f(T^p u) \overline{f}(u) d\rho(u) = \frac{1}{r} \widehat{\alpha}_f(p) \end{aligned}$$

and $\alpha_f = r\theta$. It therefore suffices to show that $\theta \ll m$.

Consider now $\widehat{\omega}(p,q)$ for $p \neq q$. In this case, $\tau^*_{q-p} \circ \tau = \rho \times \rho$ and we get

$$\begin{aligned} \widehat{\omega}(p,q) &= \int E_{\tau_{q-p}} (T^p f)(y,z) E_\tau \overline{f}(y,z) d\nu(y) d\lambda(z) \\ &= \int f(T^p u) \overline{f}(v) d\tau^*_{q-p} \circ \tau(u,v) \\ &= \int f(T^p u) d\rho(u) \cdot \int \overline{f}(v) d\rho(v) = 0 \,. \end{aligned}$$

It follows that ω is invariant under the maps $(s,t) \mapsto (s+u, t-u)$ of $\mathbf{T}^2$, for every $u \in \mathbf{T}$, and therefore its two natural projections onto $\mathbf{T}$ are invariant under all translations of $\mathbf{T}$. This means that these projections are constant multiples of m. Since ω is absolutely continuous with respect to a product of two measures on $\mathbf{T}$ (e.g. the product of the maximal spectral types of (Y,ν,T) and (Z,λ,T)), we deduce that ω is absolutely continuous with respect to the product of its two natural projections (an exercise). Combining these results we get $\omega \ll m \times m$, and therefore, finally $\theta \ll m$. ∎

THEOREM 4: *If (X,μ,T) is simple weakly mixing and does not admit a non-trivial factor with absolutely continuous spectral type, then (X,μ,T) is simple of all orders.*

Proof: By induction using theorem 3. ∎

Remarks: 1. Theorem 2 of [H] states that every pairwise independent joining of $r \geq 3$ weakly mixing systems with purely singular spectrum is independent. Of course this implies that a weakly mixing, simple system with purely singular spectral type is simple of all orders, a fact which follows from theorem 4 as well.

2. A system (X,μ,T) is *rigid* if there exists a sequence $n_k \nearrow \infty$ such that $\lim(T^{n_k}A \cap A) = \mu(A)$ for every measurable subset A of X [F–W]. This clearly implies that the maximal spectral type of (X,μ,T) is singular. Thus every w.m. rigid simple system (such as the one described in [J–R,2]) is simple of all orders. (This can be deduced directly from lemma 2).

5. Simplicity of Higher Orders

THEOREM 5: *A weakly mixing system which is simple of order 3 is simple of all orders.*

LEMMA 3: *Let (X,μ,T) be w.m. and simple of order 3, (X',μ',T) a copy of (X,μ,T) and (Y,ν,T) an ergodic system. Then a pairwise independent ergodic joining of these three systems is necessarily independent.*

Proof: Let σ be a pairwise independent ergodic joining of X', X and Y such that $\sigma \neq \mu' \times \mu \times \nu$. Let $n \neq 0$ and let $\sigma_n = (id \times T^n \times id)\,\sigma$. We consider the joining $\omega = \sigma \underset{X\times Y}{\times} \sigma_n$; ω is a measure on $X' \times X \times Y \times X''$ where X'' is another copy of X. The projection of ω on $X' \times X''$ is $\sigma_n^* \circ \sigma$ which by lemma 2 is equal to $\mu' \times \mu''$. It follows that the projection of ω onto $X' \times X \times X''$ is pairwise independent and therefore, by assumption, is independent. Thus for bounded functions f, g, h on X, X' and X'' respectively

$$\int f(x)d\mu(x) \int g(x')d\mu'(x') \int h(x'')d\mu''(x'')$$
$$= \int f(x)g(x')h(x'')d\omega(x',x,y,x'')$$
$$= \int E_\sigma g(x,y) E_\sigma h\left(T^n x, y\right) f(x) d\mu(x) d\nu(y) \; .$$

In particular for $n \neq 0$ and every bounded function h in $L^2_0(X'',\mu'')$ and bounded function u on X we get, by taking $f(x) = u(T^n x)\overline{u}(x)$ and $g = \overline{h}$

$$0 = \int E_\sigma h(T^n x, y) \ u(T^n x) E_\sigma \overline{h}(x,y)\overline{u}(x) d\mu(x) d\nu(y) .$$

Summing from 1 to N

$$0 = \int \frac{1}{N} \sum E_\sigma h(T^n x, y) u(T^n x) E_\sigma \overline{h}(x,y)\overline{u}(x) d\mu(x) d\nu(y) ,$$

and by the ergodic theorem

$$0 = \int \left| \int E_\sigma h(x,y) u(x) d\mu(x) \right|^2 d\nu(y) .$$

Thus $\int E_\sigma h(x,y)u(x)d\mu(x) = 0$ ν-a.e. and for every bounded v on Y we have

$$\int E_\sigma h(x,y)u(x)v(x)d\mu(x)d\nu(y) = \int h(x')u(x)v(y)d\sigma(x',x,y) = 0.$$

This means that σ is independent contradicting our assumption. ∎

Proof of Theorem 5: Suppose (X,μ,T) is simple of order $k \geq 3$; we shall show it is simple of order $k+1$. Let σ be an ergodic joining of $k+1$ copies of X denoted $X_1,\ldots,X_{k+1}$. If for some $i \neq j$ the projection of σ on $X_i \times X_j$ is an off-diagonal, then by an induction hypotheses, σ is a product of off-diagonals and we are done. Thus we may assume that σ is pairwise independent. By the induction hypothesis the projections of σ on $X_1 \times \cdots \times X_k$ and $X_2 \times \cdots \times X_{k+1}$ are the independent joinings. Hence σ can be viewed as a pairwise independent joining of X_1, $X_2 \times \cdots \times X_k$ and X_{k+1}. Lemma 3 applies and we conclude that σ is independent. This completes the proof. ∎

Remarks: 1. Using the same method by which lemma 3 was proved, one can show that when (X,μ,T) is w.m. and simple of order 3 and $(Y,\nu,T),(Z,\lambda,T)$ are w.m. systems then every pairwise independent ergodic joining of the three systems is independent.

2. A simple argument of relative disjointness analogous to that of corollary 1, can be used to generalize some of the preceding results to distal extensions of simple systems. Let (X,μ,T) be an ergodic distal extension of a simple system. Then for this system the assertions of corollary 1, theorem 3 and lemma 2 are valid.

If furthermore (X,μ,T) has the property that every ergodic pairwise independent 3-fold self-joining is already independent, then every pairwise independent k-fold self-joining of (X,μ,T) is independent ($k \geq 3$).

References

[Fu] H. Furstenberg, *Disjointness in ergodic theory, minimal sets and a problem in diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.

[F–W] H. Furstenberg and B. Weiss, *The finite multipliers of infinite transformation*, Springer-Verlag Lecture Notes in Math. **688** (1978), 127–132.

[H] B. Host, *Mixing of all orders and pairwise independent joinings of systems with singular spectrum*, Israel J. Math. **76** (1991), 289–298.

[J–R,1] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory and Dynamic Systems **7** (1987), 531–557.

[J–R,2] A. del Junco and D. Rudolph, *A rank-one rigid simple prime map*, Ergodic Theory and Dynamic Systems **7** (1987), 229–247.

[K] J. King, *Ergodic properties where order 4 implies infinite order*, Israel J. Math., to appear.

[R] D. Rudolph, *An example of a measure-preserving map with minimal selfjoinings and applications*, J. Analyse Math. **35** (1979), 97–122.

[V] W. A. Veech, *A criterion for a process to be prime*, Monatshefte Math. **94** (1982), 335–341.